MAX PLANCK GESELLSCHAFT
MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

Cross Approximation for Matrices and Tensors

CSC Reading Group

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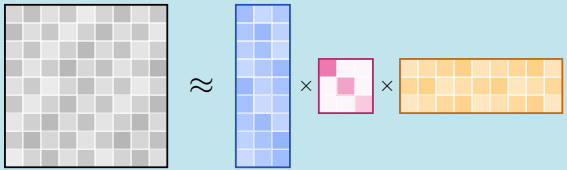
Organization

1. Motivation and notation
2. CUR/cross prototype algorithm
3. Good submatrix selection
4. Practical CUR/cross approximation algorithms
5. Connections to MOR with DEIM and tensor cross

What is and why cross approximation?

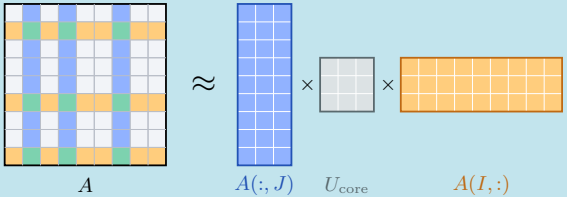
Singular value decomposition

$$A \approx A_r := U_r \Sigma_r V_r^\top$$



Cross/CUR approximation

$$A \approx C U_{\text{core}} R$$



	SVD	Cross/CUR
What it gives	Optimal dense low-rank approximation	Sample-based low-rank approximation
Advantages	Optimal error and robust standard algorithms	Interpretability, entry-query access, and tensor-friendly
Disadvantages	Loses original structure; can be expensive	Hard index selection and stability issues
When to use	Full matrix access; dense factors OK	Expensive entries; meaningful rows/columns/fibers

The cross/CUR approximation problem

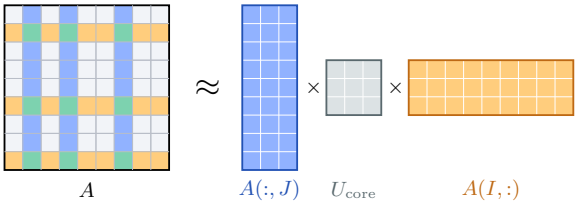
Problem formulation

Given a matrix $A \in \mathbb{R}^{m \times n}$, target rank r, solve $\min_{\substack{|I|=r,\ |J|=r \\ U_{\text{core}} \in \mathbb{R}^{r \times r}}} \|A - A(:, J) U_{\text{core}} A(I, :)\|_{\text{F}}$

Two main questions:

1. How to select the rows and columns (index sets I, J)?
2. How to choose the core U_{core}?

1. is very hard; 2. is easy once 1. is fixed.

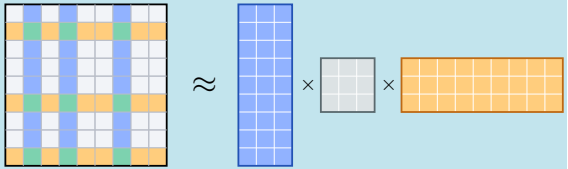


- For fixed I, J, the optimal core is $U_{\text{core}} = A(:, J)^{\dagger} A A(I, :)^{\dagger} \rightsquigarrow$ CUR approximation
- Another popular choice is $U_{\text{core}} = A(I, J)^{-1} \rightsquigarrow$ cross approximation

Two choices of the core: CUR and cross

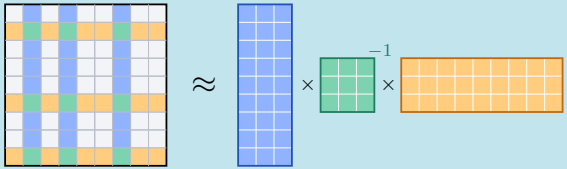
CUR (pseudoskeleton) approximation

$$A_{\mathrm{CUR}} := A(:, J)A(:, J)^{\dagger}AA(I, :)^{\dagger}A(I, :)$$



Cross (skeleton, interpolatory) approximation

$$A_{\mathrm{cross}} := A(:, J)A(I, J)^{-1}A(I, :)$$



	CUR	Cross
Core choice	Optimal for fixed I, J	Interpolatory: $A_{\mathrm{cross}}(:, J) = A(:, J)$, $A_{\mathrm{cross}}(I, :) = A(I, :)$
Core formation	Needs the whole matrix A	Only needs the intersection $A(I, J)$
Existence result: $\exists I, J$ s.t.	$\|A - A_{\mathrm{CUR}}\|_{\mathrm{F}} \leq \sqrt{2r+2}\ \|A - A_r\|_{\mathrm{F}}$	$\|A - A_{\mathrm{cross}}\|_{\mathrm{F}} \leq (r+1)\ \|A - A_r\|_{\mathrm{F}}$



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From CUR/cross to two related problems

CUR approximation

$$\min_{|I|=r,\ |J|=r} \left\| A - \underbrace{A(:,J)A(:,J)^\dagger}_{:=\Pi_J} A \underbrace{A(I,:)^\dagger A(I,:)}_{:=\Pi_I} \right\|_{\mathrm{F}}$$

- Need selected J s.t. $\mathrm{col}(A(:,J)) \approx \mathrm{col}(A)$
- This is the **column subset selection (CSSP)**:

$$\min_{|J|=r} \left\| A - \underbrace{A(:,J)A(:,J)^\dagger}_{:=\Pi_J} A \right\|_{\mathrm{F}}$$

Cross approximation

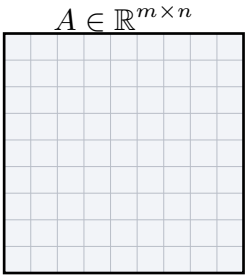
$$\min_{|I|=r,\ |J|=r} \left\| A - A(:,J)A(I,J)^{-1}A(I,:) \right\|_{\mathrm{F}}$$

- Not only need subspace approx., but also a stable intersection $A(I,J)$
- **Good submatrix selection**: for fixed J, choose I s.t.

$$\min_{|I|=r} \left\| A(I,J)^{-1} \right\|_2$$

CUR asks for good subspaces; Cross asks for good subspaces + a stable intersection.

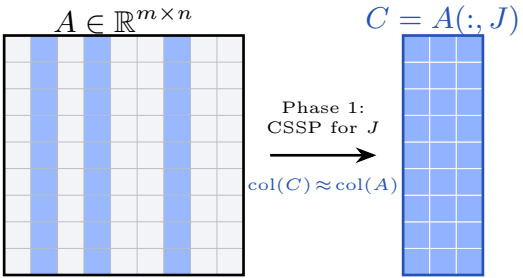
CUR/cross approximation algorithm



A typical prototype CUR/cross approximation algorithm for $A \in \mathbb{R}^{m \times n}$

1: Phase 1: choose columns J and get $C = A(:, J)$ (**CSSP step**)
2: Phase 2: choose rows I from $C = A(:, J)$ and get $R = A(I, :)$ (**Good submatrix step**)
3: Phase 3: choose a core U_{core}, e.g., $C^{\dagger}AR^{\dagger}$ for CUR or $A(I, J)^{-1}$ for cross
4: **return** I, J, U_{core}

CUR/cross approximation algorithm



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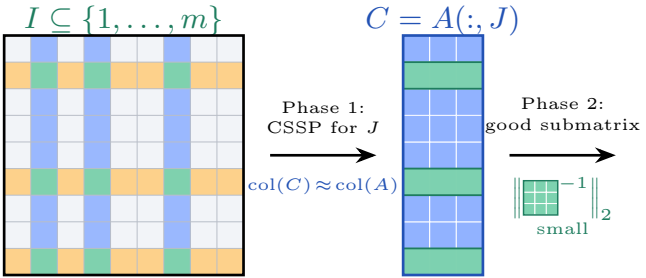

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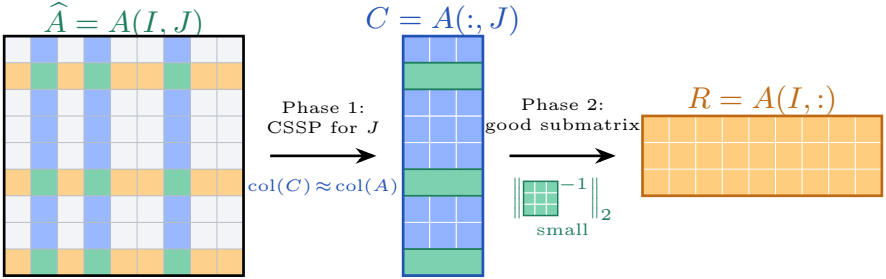

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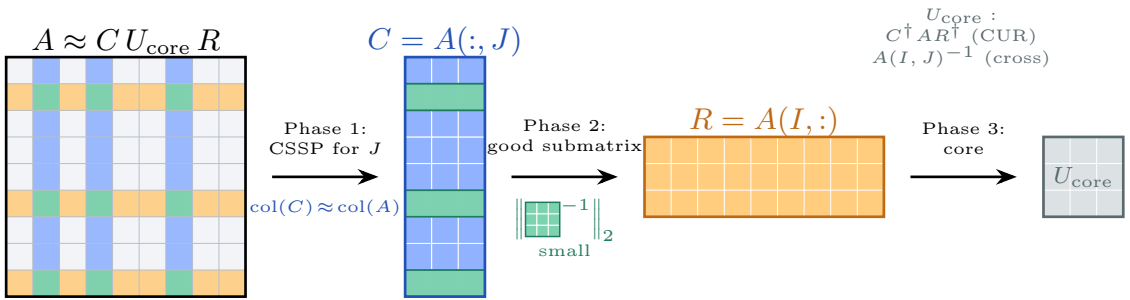
CUR/cross approximation algorithm



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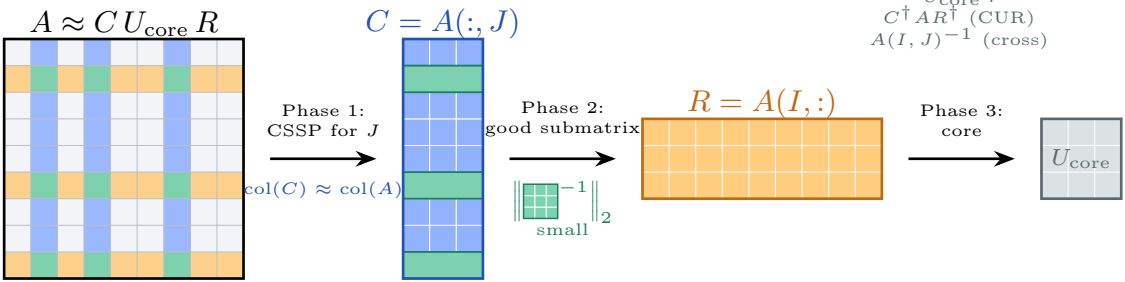
Phase 1: CSSP through a row-space basis



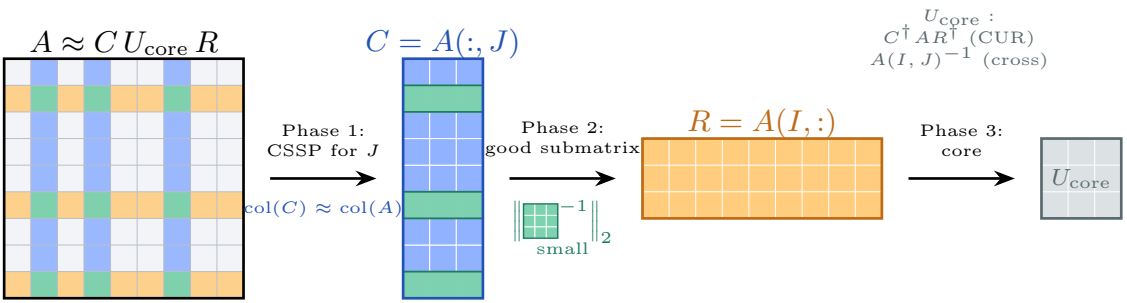
CSSP can also be done using good submatrix selection:



Using e.g.,

1. $[\sim, \sim, V^\top] = \texttt{truncated_svd}(A)$;
2. lazily, randomly select some rows $V = A(I', :)$.

Phase 1: CSSP through a row-space basis



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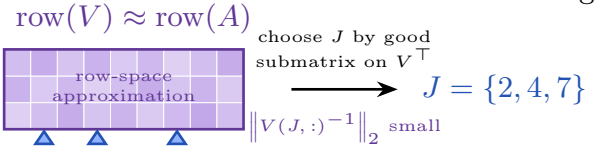

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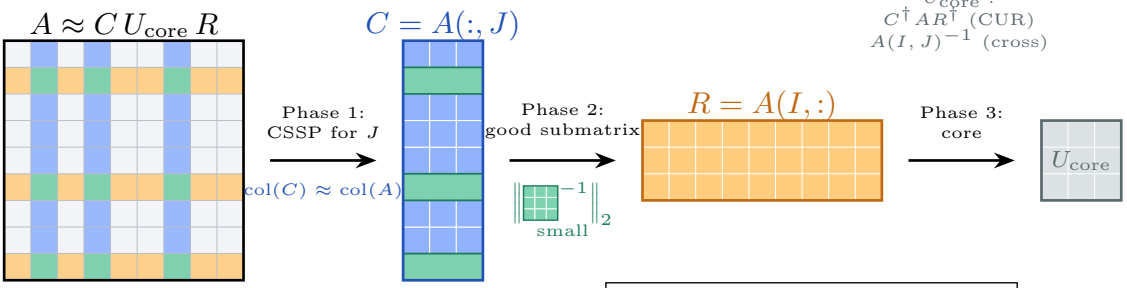
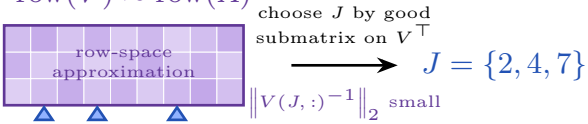


CSSP can also be done using **good submatrix selection**:



Our focus!

Organization

1. Motivation and notation
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How to select a good submatrix?

Good submatrix selection problem

Given $U \in \mathbb{R}^{n\times r},\ n > r$ with $U^\top U = \mathrm{Id}_r$, solve $\min\limits_{|I|=r} \left\|U(I,:)^{-1}\right\|_2$.

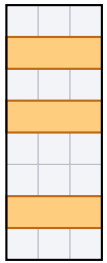
- Naive solution: compare all $\binom{n}{r}$ choices.
- In fact, this is an NP-hard problem and do not admit a PTAS.

A sufficient condition: maximum volume.

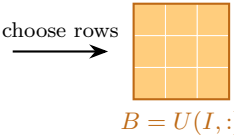
$$I \in \arg\max_{|I|=r} \mathrm{vol}(U(I,:)) \implies \left\|U(I,:)^{-1}\right\|_2 \le \sqrt{1 + r(n-r)}.$$

Here, $\mathrm{vol}(U(I,:)) := |\det U(I,:)|$.

- Maximum volume is still NP-hard.
- Practical algorithms go for greedily maximizing the volume.



$U \in \mathbb{R}^{n\times r}$



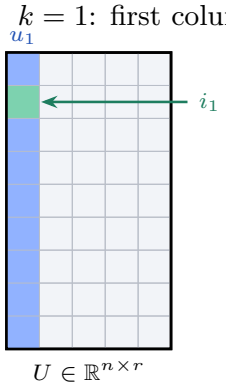
[Goreinov/Tyrtyshnikov/Zamarashkin'97], [Çivril/Magdon-Ismail'09], [Goreinov et el'10].

Forward greedy I: add one element at a time

Process column by column

- Step $k = 1$ (first column):
  Select a 1×1 submatrix s.t. the volume is maximized:

$$i_1 = \arg\max_i |\det U(i,1)| = \arg\max_i |u_{i1}| \rightsquigarrow I_1 = \{i_1\}.$$



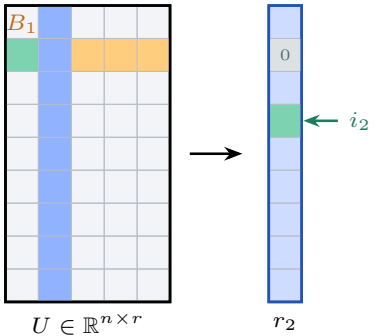
Forward greedy I: add one element at a time

Process column by column

- Step $k = 1$ (first column): $I_1 = \{\arg\max_i |u_{i1}|\}$.
- Step $k = 2$ (second column):
  Select a 2×2 submatrix containing the first one s.t. the volume is maximized:

$$
\begin{aligned}
i_2 &= \arg\max_i |\det U(I_1 \cup \{i\}, 1:2)| \\
&= \arg\max_i \left| \det \begin{pmatrix} u_{i_1,1} & u_{i_1,2} \\ u_{i,1} & u_{i,2} \end{pmatrix} \right| \\
&= \arg\max_i |u_{i_1,1}(u_{i,2} - u_{i,1}\frac{u_{i_1,2}}{u_{i_1,1}})| \\
&= \arg\max_i |u_{i,2} - u_{i,1}\frac{u_{i_1,2}}{u_{i_1,1}}| \\
&= \arg\max_i |r_2(i)|.
\end{aligned}
$$

$k = 2$: residualize column 2



$$r_2 = U(:,2) - U(:,1)B_1^{-1}U(I_1,2)$$

Forward greedy I: add one element at a time

Process column by column

- Step $k = 1$: $I_1 = \{\arg\max_i |u_{i1}|\}$.
- Step $k = 2$: $I_2 = I_1 \cup \{\arg\max_i |r_2(i)|\}$.
- Suppose we have selected I_{k-1} for the first $k-1$ columns and $B_{k-1} = U(I_{k-1}, 1:k-1)$.
- Step k (k-th column): Select a $k \times k$ submatrix containing the previous one s.t. the volume is maximized:

$$
\begin{aligned}
i_k &= \arg\max_i |\det U(I_{k-1} \cup \{i\}, 1:k)| \\
&= \arg\max_i \left| \det \begin{pmatrix} B_{k-1} & U(I_{k-1}, k) \\ U(i, 1:k-1) & u_{i,k} \end{pmatrix} \right| \\
&= \arg\max_i |\det B_{k-1}| |u_{i,k} - U(i, 1:k-1) B_{k-1}^{-1} U(I_{k-1}, k)| \\
&= \arg\max_i |u_{i,k} - U(i, 1:k-1) B_{k-1}^{-1} U(I_{k-1}, k)| \\
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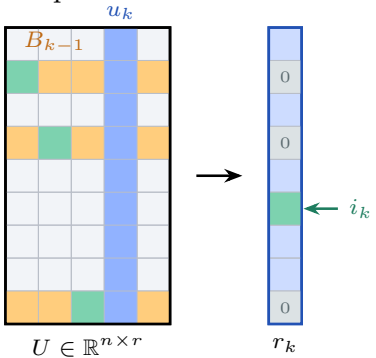
step k: residualize column k



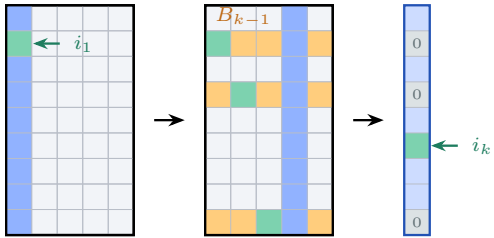
$$r_k = U(:, k) - U(:, 1:k-1) B_{k-1}^{-1} U(I_{k-1}, k)$$

Forward greedy I: add one element at a time

DEIM ([Chaturantabut/Sorensen'10])

Input: $U \in \mathbb{R}^{n\times r}$.
Output: row index set I.
$I = \{\arg\max_i |U(i,1)|\}$.
for $k = 2, \ldots, r$ **do**
    $r_k = U(:,k) - U(:,1:k-1)U(I,1:k-1)^{-1}U(I,k)$.
    $I \leftarrow I \cup \{\arg\max_i |r_k(i)|\}$.
end for

1. first column 2. old block 3. residual



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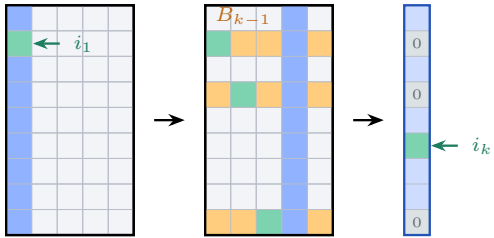
end for

LUPP (GE with partial pivoting)

Input: $U \in \mathbb{R}^{n\times r}$.
Output: row index set I.
$I = \{\arg\max_i |U(i,1)|\}$.
for $k = 2, \ldots, r$ **do**
    $U = U - \frac{1}{u_{i_{k-1},k-1}} U(:,k-1)U(i_{k-1},:)$.
    $I \leftarrow I \cup \{\arg\max_i U(i,k)\}$.
end for

1. first column 2. old block 3. residual



$r_k = U(:,k) - U(:,1:k-1)B_{k-1}^{-1}U(I_{k-1},k)$

Forward greedy I: add one element at a time

DEIM ([Chaturantabut/Sorensen'10])

Input: $U \in \mathbb{R}^{n\times r}$.
Output: row index set I.
$I = \{\arg\max_i |U(i,1)|\}$.
for $k = 2, \ldots, r$ **do**
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end for

LUPP (GE with partial pivoting)

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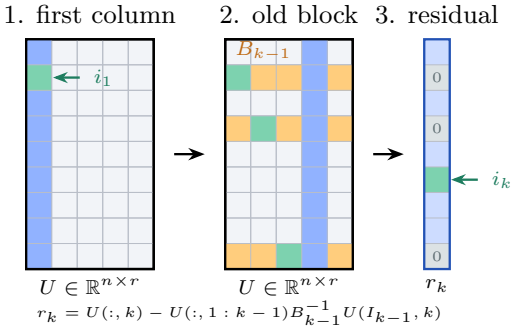
Output: row index set I.
$I = \{\arg\max_i |U(i,1)|\}$.
for $k = 2, \ldots, r$ **do**
    $U = U - \frac{1}{u_{i_{k-1},k-1}} U(:,k-1)U(i_{k-1},:)$.
    $I \leftarrow I \cup \{\arg\max_i U(i,k)\}$.
end for



Remark

- DEIM and LUPP are equivalent.
- DEIM can be viewed as a simplified LUPP where only the pivots are kept but not the LU factors.

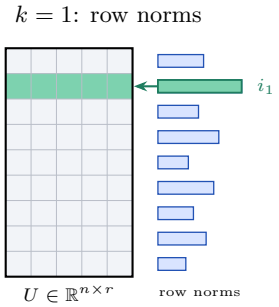
Forward greedy II: add one row at a time

Another volume-maximizing heuristic

- Step $k = 1$:
  Select a $1 \times r$ submatrix s.t. the volume is maximized:

$$i_1 = \arg\max_i \mathrm{vol}(U(i,:)) = \arg\max_i \|U(i,:)\|_2 \rightsquigarrow I_1 = \{i_1\}.$$

- Here, the volume of a rectangular matrix $M \in \mathbb{R}^{k\times r}$ is $\mathrm{vol}(M) := \sqrt{\det(MM^\top)} = \prod_{i=1}^{k} \sigma_i(M)$.



Forward greedy II: add one row at a time

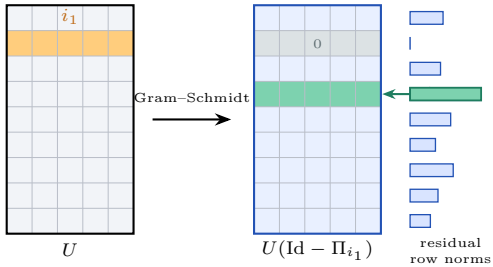
Another volume-maximizing heuristic

- Step $k = 1$: $I_1 = \{\arg\max_i \|U(i,:)\|_2\}$.
- Step $k = 2$:
  Select a $2 \times r$ submatrix containing the first one s.t. the volume is maximized:

$$
\begin{aligned}
i_2 &= \arg\max_i \operatorname{vol} \begin{bmatrix} U(i_1,:) \\ U(i,:) \end{bmatrix} \\
&= \arg\max_i \|U(i_1,:)\|_2 \left\| (\mathrm{Id} - \Pi_{i_1}) U(i,:)^\top \right\|_2 \\
&= \arg\max_i \left\| (\mathrm{Id} - \Pi_{i_1}) U(i,:)^\top \right\|_2 \\
&= \arg\max_i \left\| (\mathrm{Id} - Q_1 Q_1^\top) U(i,:)^\top \right\|_2 .
\end{aligned}
$$

Here $\Pi_{i_1} = Q_1 Q_1^\top$ with $Q_1 = U(i_1,:)^\top / \|U(i_1,:)\|_2$ is the orthogonal projection onto the first selected row direction.

$k = 2$: residual row norms



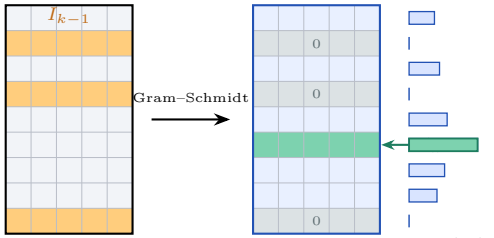
Forward greedy II: add one row at a time

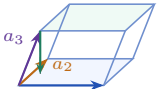
Another volume-maximizing heuristic

- Step $k = 1$: $I_1 = \{\arg\max_i \|U(i,:)\|_2\}$.
- Step $k = 2$: $I_2 = I_1 \cup \{\arg\max_i \left\|(\mathrm{Id} - Q_1 Q_1^\top)U(i,:)^\top\right\|_2\}$.
- Suppose we have selected I_{k-1} and $Q_{k-1} = \mathrm{orth}\{U(I_{k-1},:)\}$.
- Step k:
  Select a $k \times r$ submatrix containing the previous one s.t. the volume is maximized:

$$
\begin{aligned}
i_k &= \arg\max_i \mathrm{vol}\begin{bmatrix} U(I_{k-1},:) \\ U(i,:) \end{bmatrix} \\
&= \arg\max_i \mathrm{vol}(U(I_{k-1},:)) \left\|(\mathrm{Id} - \Pi_{I_{k-1}})U(i,:)^\top\right\|_2 \\
&= \arg\max_i \left\|(\mathrm{Id} - Q_{k-1}Q_{k-1}^\top)U(i,:)^\top\right\|_2 .
\end{aligned}
$$

step k: residual row norms



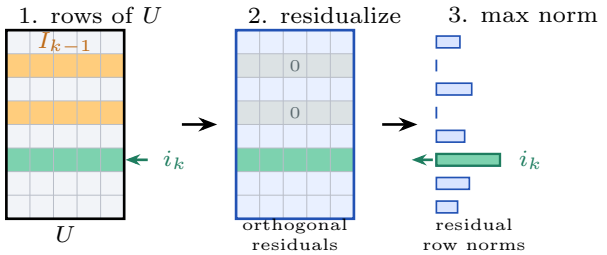


height: $(\mathrm{Id} - \Pi_{a_1,a_2})a_3$
$\mathrm{vol}([a_1, a_2, a_3])$
$= \mathrm{vol}([a_1, a_2]) \|(\mathrm{Id} - \Pi_{a_1,a_2})a_3\|_2$

Forward greedy II: add one row at a time

QRCP ([Businger/Golub'65]) on $U^\top$

Input: $U \in \mathbb{R}^{n\times r}$ with $U^\top U = \mathrm{Id}$.
Output: row index set I.
$I \leftarrow \emptyset$, $Q \leftarrow [\,]$.
for $k = 1, \ldots, r$ **do**
  $i_k \in \arg\max_i \left\|(\mathrm{Id} - QQ^\top)U(i,:)^\top\right\|_2$.
  $I \leftarrow I \cup \{i_k\}$.
  $Q \leftarrow \mathrm{orth}(Q, U(i_k,:)^\top)$.
end for



Q-DEIM ([Drmac/Gugercin'16])

Input: $U \in \mathbb{R}^{n\times r}$ with $U^\top U = \mathrm{Id}$.
Output: row index set I.
$[\sim, \sim, I] = \mathtt{qr}(U^\top, \mathtt{'vector'})$;
$I = I(1:r)$.

Remark

- Q-DEIM is QRCP applied to $U^\top$.

Why forward greedy is not enough

Limitation

Forward greedy methods work really well in practice, but their worst-case error bounds are exponential.

Recall that

$$I \in \operatorname*{arg\,max}_{|I|=r} \operatorname{vol}(U(I,:)) \implies \left\|U(I,:)^{-1}\right\|_2 \leq \sqrt{1 + r(n-r)}.$$

Theorem (Higham'02)

Let I be the index set returned by LUPP/DEIM applied to $U \in \mathbb{R}^{n\times r}$, then

$$\left\|U(I,:)^{-1}\right\|_2 \leq \sqrt{nr}\,2^{r-1}.$$

Theorem (Drmac/Gugercin'16)

Let I be the index set returned by Q-DEIM applied to $U \in \mathbb{R}^{n\times r}$ (QRCP applied to $U^\top$), then

$$\left\|U(I,:)^{-1}\right\|_2 \leq \frac{\sqrt{n-r+1}}{3}\sqrt{4^r + 6r - 1}.$$

This motivates methods that can achieve better bounds such as `maxvol` and sRRQR.

Greedy maximum volume algorithms

A sufficient condition for a good submatrix: maximum volume.

$$I \in \underset{|I|=r}{\arg\max}\, \mathrm{vol}(U(I,:)) \implies \left\|U(I,:)^{-1}\right\|_2 \leq \sqrt{1 + r(n-r)}.$$



Algorithms that greedily maximize the volume

Forward greedy	**Exchange greedy**
$\{i_1\} \rightsquigarrow \{i_1, i_2\} \rightsquigarrow \cdots \rightsquigarrow I$	$\{i_1, \ldots, i_k, \ldots, i_r\} \rightsquigarrow \{i_1, \ldots, \tilde{i}_k, \ldots, i_r\} \rightsquigarrow \cdots \rightsquigarrow I$
LUPP/DEIM, QRCP/Q-DEIM	maxvol/sRRQR

Exchange greedy: maxvol

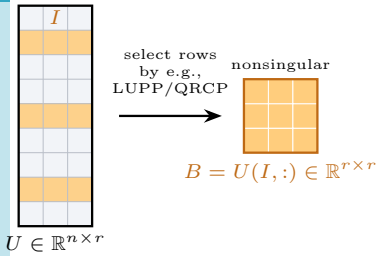
Exchange one row at a time to increase the volume

- Step 0: initialize the index set I.
- It starts from a nonsingular $r \times r$ submatrix.

$$B = U(I, :) \in \mathbb{R}^{r \times r},$$

- The initialization can be done using e.g., LUPP or QRCP as discussed.
- The question is whether an unselected row can increase the volume.
- WLOG, assume $I = \{1, \ldots, r\}$.

start from a full block



Exchange greedy: maxvol

Exchange one row at a time to increase the volume

- Step 0: initialize the index set I by e.g. applying LUPP or QRCP.
- WLOG, assume $I = \{1, \dots, r\}$ and $I_c = \{1, \dots, n\} \setminus I$.
- Consider B_{new} being B with row i replaced by $U(\alpha, :)$, $\alpha \in I_c$.
- The exchange should increase the volume as much as possible:

$$(\alpha, i) \in \arg\max_{\alpha, i} \operatorname{vol}(B_{\text{new}}) / \operatorname{vol}(B).$$

Let's compute the new volume:

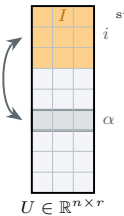
$$\det(B_{\text{new}}) = \det \begin{pmatrix} B(1,:) \\ \vdots \\ U(\alpha,:) \\ \vdots \\ B(r,:) \end{pmatrix} = \sum_{\ell=1}^{r} t_{\alpha\ell} \det \begin{pmatrix} B(1,:) \\ \vdots \\ B(\ell,:) \\ \vdots \\ B(r,:) \end{pmatrix} = t_{\alpha i} \det B$$

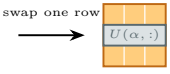
Here $U(\alpha, :) = t_\alpha^\top B$. Therefore,

$$\operatorname{vol}(B_{\text{new}}) / \operatorname{vol}(B) = |t_{\alpha i}|.$$

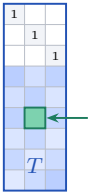
$t_\alpha^\top = U(\alpha, :) B^{-1} \rightsquigarrow t_{\alpha i}$ is the entry of $U(I_c, :) B^{-1}$.

one candidate exchange





$$\frac{\operatorname{vol}(B_{\text{new}})}{\operatorname{vol}(B)} = |t_{\alpha,i}|$$



$$= UU(I,:)^{-1} = \begin{bmatrix} \mathrm{Id}_r \\ T \end{bmatrix}$$

Exchange greedy: maxvol

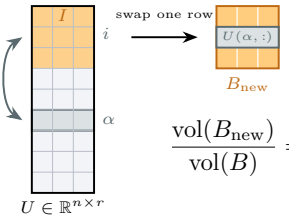
Exchange one row at a time to increase the volume

- Step 0: initialize the index set I by e.g. applying LUPP or QRCP. Reorder I so that $I = \{1, \dots, r\}$.
- Step 1: compute the $UU(I,:)^{-1} = \begin{bmatrix} \mathrm{Id}_r \\ T \end{bmatrix}$ matrix.
- Step 2: find the largest entry $t_{\alpha i}$ of T.
- Step 3: If $|t_{\alpha i}| > 1$, the volume is increased by $|t_{\alpha i}|$, exchange i and α; otherwise, return.

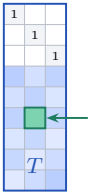
Practically, stopping criterion: $|t_{\alpha i}| \leq \mu$ for $\mu > 1$.

$$\begin{aligned} \|U(I,:)^{-1}\|_2 = \|UU(I,:)^{-1}\|_2 &= \left\| \begin{bmatrix} \mathrm{Id}_r & T \end{bmatrix}^\top \right\|_2 \\ &= \sqrt{\lambda_{\max}(\mathrm{Id}_r^2 + T^2)} = \sqrt{1 + \lambda_{\max}(T^2)} \\ &= \sqrt{1 + \|T\|_2^2} \leq \sqrt{1 + \|T\|_{\mathrm{F}}^2} \leq \sqrt{1 + r(n-r)\mu^2}. \end{aligned}$$

one candidate exchange



$$\frac{\mathrm{vol}(B_{\mathrm{new}})}{\mathrm{vol}(B)} = |t_{\alpha,i}|$$



$$= UU(I,:)^{-1} = \begin{bmatrix} \mathrm{Id}_r \\ T \end{bmatrix}$$

maxvol algorithm

The maxvol algorithm with threshold $\mu \geq 1$

Input: $U \in \mathbb{R}^{n \times r}$, threshold $\mu \geq 1$.
Output: row index set I.
Apply LUPP on U (or QRCP on $U^\top$) to get I.
$I_c \leftarrow \{1, \ldots, n\} \setminus I$.
repeat
  $B \leftarrow U(I,:)$, $T \leftarrow U(I_c,:)B^{-1}$.
  $(\alpha, i) = \arg\max_{\alpha,i} |T(\alpha, i)|$.
  if $|T(\alpha, i)| \leq \mu$ **then**
    stop.
  else
    $\alpha \leftarrow I_c(\alpha)$, $I(i) \leftarrow \alpha$.
  end if
until stopped

Theorem (Goreinov et al.'10)

Let I be the index set returned by maxvol with threshold μ. Then

$$\|U(I,:)^{-1}\|_2 \leq \sqrt{1 + \mu^2 r(n-r)}.$$

Remark

- Each entry of T is a one-row volume ratio.
- maxvol searches for an exchange that increases the volume by at least μ.

Goreinov et al.'10 = Goreinov, S. A., Oseledets, I. V., Savostyanov, D. V., Tyrtyshnikov, E. E., & Zamarashkin, N. L. (2010): How to find a good submatrix. In Matrix Methods: Theory, Algorithms and Applications, 247–256.

`maxvol` is closely related to sRRQR

Apply `maxvol` to U with threshold μ we get

$$P^\top U = \begin{bmatrix} B \\ U_c \end{bmatrix} \text{ s.t. } \|T\|_{\max} = \|U_c B^{-1}\|_{\max} \leq \mu.$$

Apply sRRQR ([Gu/Eisenstat'96]) to $U^\top$

$$U^\top \Pi = Q \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} \text{ s.t. } \|R_{11}^{-1} R_{12}\|_{\max} \leq \mu.$$

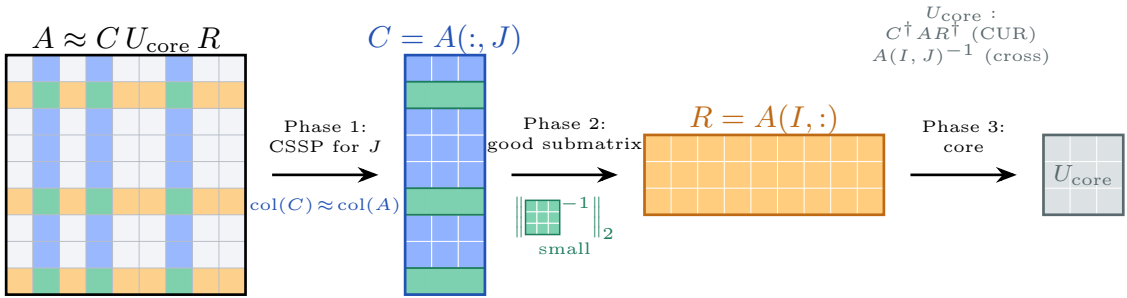
- `maxvol` controls $\|T\|_{\max} \leq \mu$.
- sRRQR controls $\|R_{11}^{-1} R_{12}\|_{\max} \leq \mu$.
- One can show: $R_{11}^{-1} R_{12} = T^\top$.
- `maxvol` usually swaps using the largest entry of T.
- Gu–Eisenstat sRRQR only needs to find a violating entry with $|(R_{11}^{-1} R_{12})_{ij}| > \mu$.



Organization

1. Motivation and notation
2. CUR/cross prototype algorithm
3. Good submatrix selection
4. Practical CUR/cross approximation algorithms
5. Connections to MOR with DEIM and tensor cross

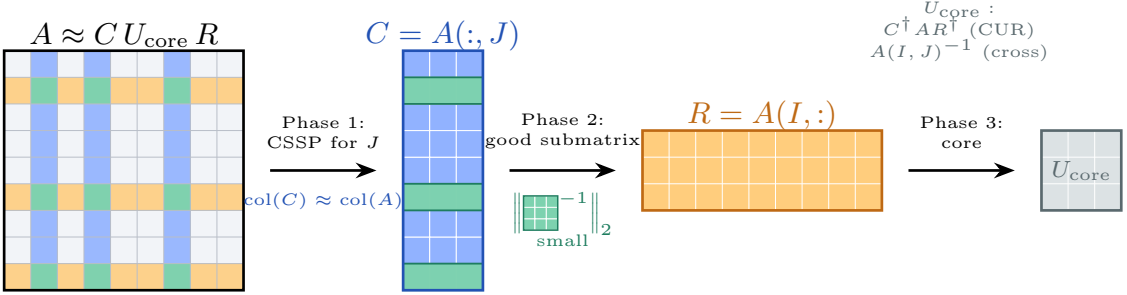
CUR/cross approximation algorithm



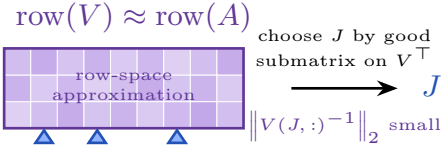
A typical prototype CUR/cross approximation algorithm for $A \in \mathbb{R}^{m \times n}$

1: Phase 1: choose columns J and get $C = A(:, J)$ (**CSSP step**)
2: Phase 2: choose rows I from $C = A(:, J)$ and get $R = A(I, :)$ (**Good submatrix step**)
3: Phase 3: choose a core U_{core}, e.g., $C^\dagger A R^\dagger$ for CUR or $A(I, J)^{-1}$ for cross
4: **return** I, J, U_{core}

Phase 1: CSSP through a row-space basis



CSSP can also be done using good submatrix selection:



1. choose a row-space approximator:
   a. $[\sim, \sim, V^\top] =$ `truncated_svd`(A);
   b. lazily select rows $V = A(I', :)$.
2. choose a good submatrix selector:
   LUPP / QRCP / maxvol

Practical algorithm 1: basis plus selector

$2\times$ basis plus selector

1: **Input:** matrix $A \in \mathbb{R}^{m\times n}$, target rank r, and a `good-submatrix` selector.
2: Compute column-space and row-space bases $U \in \mathbb{R}^{m\times r}$ and $V \in \mathbb{R}^{r\times n}$.
3: E.g., use $A \approx U\Sigma V^\top$ from a truncated or randomized SVD.
4: $J \leftarrow$ `good-submatrix`$(V^\top)$.
5: $I \leftarrow$ `good-submatrix`(U).
6: $C \leftarrow A(:, J)$ and $R \leftarrow A(I, :)$.
7: Choose U_{core}, e.g., $C^\dagger A R^\dagger$ for CUR or $A(I, J)^{-1}$ for cross.

`good-submatrix` can be LUPP, QRCP, `maxvol`, or sRRQR.

Sorensen, D. C., & Embree, M. (2016): A DEIM induced CUR factorization. SIAM Journal on Scientific Computing, 38(3), A1454–A1482.

Practical algorithm 2: sketch then pivot

$1\times$ basis plus selector + good-submatrix on selected cols

1: **Input:** matrix $A \in \mathbb{R}^{m\times n}$, target rank r, and a `good-submatrix` selector.
2: Compute a row-space basis $V \in \mathbb{R}^{r\times n}$.
3: E.g., use $A \approx U\Sigma V^\top$ from a truncated or randomized SVD.
4: More easily, $V \leftarrow \Omega A$ with $\Omega \in \mathbb{R}^{r\times m}$.
5: $J \leftarrow$ `good-submatrix`$(V^\top)$.
6: $C \leftarrow A(:, J)$.
7: $I \leftarrow$ `good-submatrix`(C).
8: $R \leftarrow A(I, :)$
9: Choose U_{core}, e.g., $C^\dagger A R^\dagger$ for CUR or $A(I, J)^{-1}$ for cross.

Both algorithms still requires the full matrix, i.e., **linear complexity** $\mathcal{O}(mn)$ ($\mathcal{O}(n^d)$ in tensors).

Dong, Y., & Martinsson, P. G. (2023). Simpler is better: a comparative study of randomized pivoting algorithms for CUR and interpolative decompositions. Advances in Computational Mathematics, 49(4), 66.

Practical algorithm 3: matrix cross

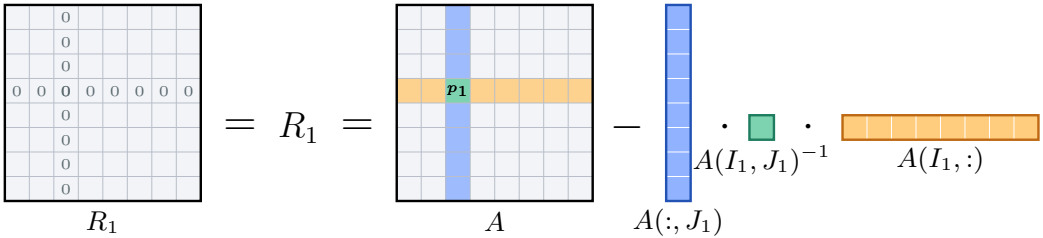
Alternating good-submatrix refinement

Input: matrix $A \in \mathbb{R}^{m \times n}$, target rank r, and an initial set J (usually random or just $\{1, \cdots, r\}$).
repeat
    Form $C \leftarrow A(:, J)$.
    Compute an orthornormal basis $Q_C = \mathrm{orth}(C)$.
    $I \leftarrow \texttt{good-submatrix}(Q_C)$.
    Form $R \leftarrow A(I, :)$.
    Compute an orthornormal basis $Q_C = \mathrm{orth}(R^\top)$.
    $J \leftarrow \texttt{good-submatrix}(Q_R)$.
until the indices or the approximation stop changing
Choose $U_{\mathrm{core}} = A(I, J)^{-1}$ or a stabilized core $A(I, J)^\dagger$.

This is the matrix algorithm behind many tensor-cross routines, as it only requires $\mathcal{O}((m+n)r)$ entries of the matrix ($\mathcal{O}(dn\,\mathrm{poly}(r))$ in tensors), i.e., **sublinear complexity**.

[Oseledets/Tyrtyshnikov'10], [Savostyanov'14], [Chiu/Demanet'13], [Li/Yang/Martin/Ho/Ying'15], [Cortinovis/Ying'26].

Practical algorithm 4: ACA



$(i_1, j_1) \in \arg\max_{i,j} |A(i,j)|$

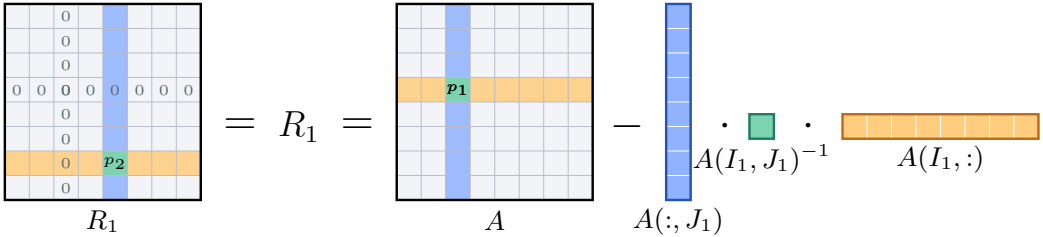
Adaptive cross approximation ([Bebendorf'00], [Bebendorf'11])

1: **Input:** entry oracle for $A \in \mathbb{R}^{m \times n}$ and tolerance ε.
2: $R_0 \leftarrow A$, $I \leftarrow \emptyset$, and $J \leftarrow \emptyset$.
3: **for** $k = 1, 2, \ldots$ **do**
4: $\quad (i_k, j_k) \in \arg\max_{i,j} |R_{k-1}(i,j)|$; $p_k \leftarrow R_{k-1}(i_k, j_k)$.
5: $\quad$ **if** $|p_k| \leq \varepsilon$ **then** stop.
6: $\quad I \leftarrow I \cup \{i_k\}$ and $J \leftarrow J \cup \{j_k\}$; $R_k \leftarrow R_{k-1} - R_{k-1}(:, j_k) p_k^{-1} R_{k-1}(i_k, :)^\top$.
7: **end for**
8: **return** $A(:, J)$, $A(I, J)^{-1}$, and $A(I, :)$.

Remark

ACA with complete pivoting (partial pivoting) is just GECP (GEPP) directly applied to A.

Practical algorithm 4: ACA



$(i_2, j_2) \in \arg\max_{i,j} |R_1(i, j)|$

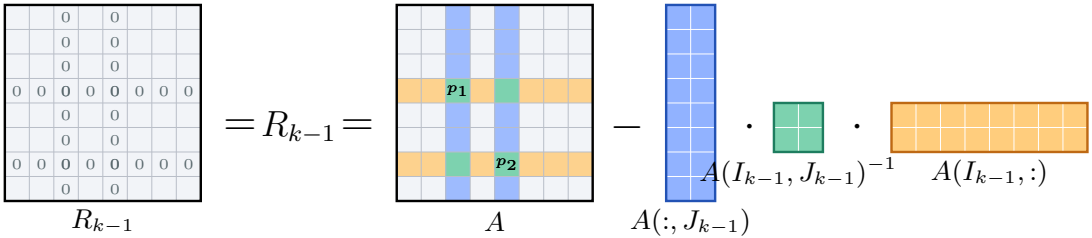
Adaptive cross approximation ([Bebendorf'00], [Bebendorf'11])

1: **Input:** entry oracle for $A \in \mathbb{R}^{m \times n}$ and tolerance ε.
2: $R_0 \leftarrow A$, $I \leftarrow \emptyset$, and $J \leftarrow \emptyset$.
3: **for** $k = 1, 2, \ldots$ **do**
4: $\quad (i_k, j_k) \in \arg\max_{i,j} |R_{k-1}(i, j)|$; $p_k \leftarrow R_{k-1}(i_k, j_k)$.
5: $\quad$ **if** $|p_k| \leq \varepsilon$ **then** stop.
6: $\quad I \leftarrow I \cup \{i_k\}$ and $J \leftarrow J \cup \{j_k\}$; $R_k \leftarrow R_{k-1} - R_{k-1}(:, j_k) p_k^{-1} R_{k-1}(i_k, :)^\top$.
7: **end for**
8: **return** $A(:, J)$, $A(I, J)^{-1}$, and $A(I, :)$.

Remark

ACA with complete pivoting (partial pivoting) is just GECP (GEPP) directly applied to A.

Practical algorithm 4: ACA



$$R_{k-1} = A - A(:, J_{k-1})A(I_{k-1}, J_{k-1})^{-1}A(I_{k-1}, :)$$

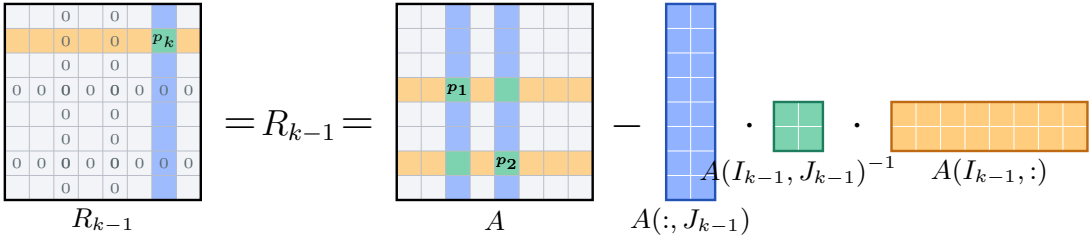
Adaptive cross approximation ([Bebendorf'00], [Bebendorf'11])

1: **Input:** entry oracle for $A \in \mathbb{R}^{m \times n}$ and tolerance ε.
2: $R_0 \leftarrow A$, $I \leftarrow \emptyset$, and $J \leftarrow \emptyset$.
3: **for** $k = 1, 2, \ldots$ **do**
4: $\quad (i_k, j_k) \in \arg\max_{i,j} |R_{k-1}(i, j)|$; $p_k \leftarrow R_{k-1}(i_k, j_k)$.
5: $\quad$ **if** $|p_k| \leq \varepsilon$ **then** stop.
6: $\quad I \leftarrow I \cup \{i_k\}$ and $J \leftarrow J \cup \{j_k\}$; $R_k \leftarrow R_{k-1} - R_{k-1}(:, j_k)p_k^{-1}R_{k-1}(i_k, :)^\top$.
7: **end for**
8: **return** $A(:, J)$, $A(I, J)^{-1}$, and $A(I, :)$.

Remark

ACA with complete pivoting (partial pivoting) is just GECP (GEPP) directly applied to A.

Practical algorithm 4: ACA



$$(i_k, j_k) \in \arg\max_{i,j} |R_{k-1}(i,j)|$$

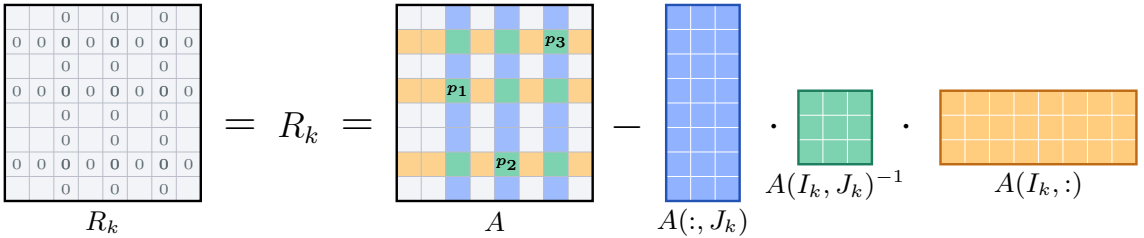
Adaptive cross approximation ([Bebendorf'00], [Bebendorf'11])

1: **Input:** entry oracle for $A \in \mathbb{R}^{m \times n}$ and tolerance ε.
2: $R_0 \leftarrow A$, $I \leftarrow \emptyset$, and $J \leftarrow \emptyset$.
3: **for** $k = 1, 2, \ldots$ **do**
4: $\quad (i_k, j_k) \in \arg\max_{i,j} |R_{k-1}(i,j)|$; $p_k \leftarrow R_{k-1}(i_k, j_k)$.
5: $\quad$ **if** $|p_k| \leq \varepsilon$ **then** stop.
6: $\quad I \leftarrow I \cup \{i_k\}$ and $J \leftarrow J \cup \{j_k\}$; $R_k \leftarrow R_{k-1} - R_{k-1}(:, j_k) p_k^{-1} R_{k-1}(i_k, :)^\top$.
7: **end for**
8: **return** $A(:, J)$, $A(I, J)^{-1}$, and $A(I, :)$.

Remark

ACA with complete pivoting (partial pivoting) is just GECP (GEPP) directly applied to A.

Practical algorithm 4: ACA



$$R_k = A - A(:, J_k)A(I_k, J_k)^{-1}A(I_k, :)$$

Adaptive cross approximation ([Bebendorf'00], [Bebendorf'11])

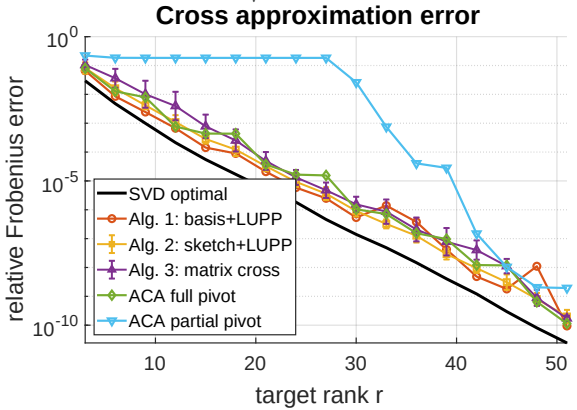
1: **Input:** entry oracle for $A \in \mathbb{R}^{m \times n}$ and tolerance ε.
2: $R_0 \leftarrow A$, $I \leftarrow \emptyset$, and $J \leftarrow \emptyset$.
3: **for** $k = 1, 2, \ldots$ **do**
4: $\quad (i_k, j_k) \in \arg\max_{i,j} |R_{k-1}(i, j)|$; $p_k \leftarrow R_{k-1}(i_k, j_k)$.
5: $\quad$ **if** $|p_k| \leq \varepsilon$ **then** stop.
6: $\quad I \leftarrow I \cup \{i_k\}$ and $J \leftarrow J \cup \{j_k\}$; $R_k \leftarrow R_{k-1} - R_{k-1}(:, j_k)p_k^{-1}R_{k-1}(i_k, :)^\top$.
7: **end for**
8: **return** $A(:, J)$, $A(I, J)^{-1}$, and $A(I, :)$.

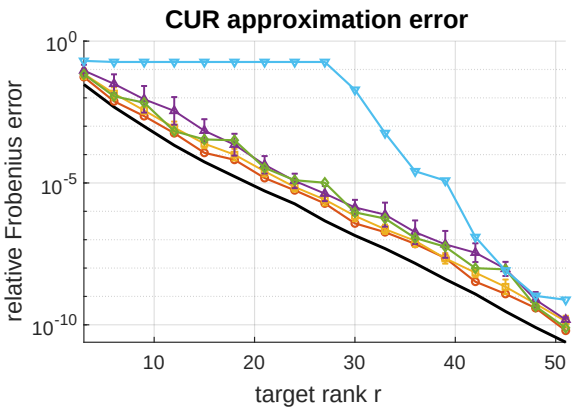
Remark

ACA with complete pivoting (partial pivoting) is just GECP (GEPP) directly applied to A.

Numerical experiment: cross/CUR approximation

$$A_{ij} = \exp\left(-15\sqrt{x_j^2 + y_i^2}\right) + \exp\left(-75\sqrt{(x_j-1)^2 + (y_i-1)^2}\right), \quad x_j \in [0,1], \quad y_i \sim \mathrm{Unif}(0,1).$$





This test follows the setup of [Cortinovis/Kressner'26, Fig. 3(b)] with $n = 2000$, and the randomized methods use 20 independent runs.

Organization

1. Motivation and notation
2. CUR/cross prototype algorithm
3. Good submatrix selection
4. Practical CUR/cross approximation algorithms
5. Connections to MOR with DEIM and tensor cross

Why should we care?

Model reduction

- DEIM selects entries of nonlinear terms.
- The online cost depends on selected entries, not on the full state dimension.

H-matrices

- ACA queries expensive matrix entries adaptively.
- Low-rank blocks are built from sampled rows and columns.

Tensor cross

- Fibers replace rows and columns.
- Good-submatrix ideas reappear on unfoldings and interfaces.

The same index-selection question reappears in many computational settings.

Vector approximation by interpolation

From orthogonal projection to oblique interpolation

Let $f \in \mathbb{R}^n$ be a high-dimensional vector; let $U \in \mathbb{R}^{n \times r}$ have orthonormal columns.
Goal: want to approximate f in the subspace $\operatorname{range} U$.
Best approximation: orthogonal projection $\Pi_U f = UU^\top f$. **Disadvantage**: Requires all entries of f.
For selected indices $I = \{i_1, \ldots, i_r\}$, define the sampling matrix

$$S_I = [e_{i_1}, e_{i_2}, \ldots, e_{i_r}].$$

DEIM replaces it by the oblique projection $\Pi_I f = U(S_I^\top U)^{-1} S_I^\top f = UU(I,:)^{-1} S_I^\top f$.

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in \mathbb{R}^n \quad \text{v.s.} \quad S_I^\top f = \begin{bmatrix} f_{i_1} \\ \vdots \\ f_{i_r} \end{bmatrix} \in \mathbb{R}^r$$

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Error amplification

$$\|f - \Pi_I f\|_2 \leq \|U(I,:)^{-1}\|_2 \cdot \|f - UU^\top f\|_2.$$

Recall: LUPP gives $\left\|U(I,:)^{-1}\right\|_2 \leq \sqrt{nr}\,2^{r-1}$, QRCP gives $\left\|U(I,:)^{-1}\right\|_2 \leq \dfrac{\sqrt{n-r+1}}{3}\sqrt{4^r + 6r - 1}$,
`maxvol` (sRRQR) gives $\left\|U(I,:)^{-1}\right\|_2 \leq \sqrt{1 + \mu^2 r(n-r)}$.

DEIM is interpolation at selected entries

Interpolation condition

The residual vanishes at the selected entries:

$$S_I^\top (I - \Pi_I) f = 0.$$

Hence $\Pi_I f$ matches f exactly on the index set I.

Same good-submatrix problem

1: The basis U is the space approximator.
2: The selected rows form $S_I^\top U = U(I, :)$.
3: The stability factor is $\|U(I, :)^{-1}\|_2$.
4: LUPP gives the original DEIM ([Chaturantabut/Sorensen'10])
5: QRCP gives Q-DEIM ([Drmac/Gugercin'16])
6: sRRQR (plus leverage score sampling) gives R-DEIM ([Drmac/Saibaba'18], [Saibaba'20]).

Continuous version: EIM ([Barrault/Maday/Nguyen/Patera'04]).

POD+DEIM

Consider a large ODE

$$\dot{x}(t) = Ax(t) + F(x(t)), \qquad A \in \mathbb{R}^{n\times n}.$$

Basic idea of Proper orthogonal decomposition (POD)

1. Simulate the ODE for one or more initial conditions and collect trajectories at discrete time points into snapshot matrix:
$$X = \big([x(t_1), \ldots, x(t_m)]\big) \in \mathbb{R}^{n\times m}.$$
2. Compute basis approximation $\mathrm{range}(U) \approx \mathrm{range}(X)$ with $U \in \mathbb{R}^{n\times r}$, $r \ll n$, e.g., $[U, \sim, \sim] = \texttt{truncated_svd}(X)$.
3. Approximate by orth. proj. $x(t) \approx UU^\top x(t) = Uy(t)$ with $y(t) = U^\top x(t)$. Galerkin projection gives a small ODE:
$$\dot{y}(t) = U^\top AUy(t) + U^\top F(Uy(t)).$$

Problem: The nonlinear term still asks for the full vector $U^\top F(Uy(t)) \in \mathbb{R}^n$.

Solution: Replace orth. proj. by oblique proj., i.e., DEIM: $U^\top F(Uy(t)) \approx S_I^\top U^{-1} S_I^\top F(Uy(t))$.

POD+DEIM

POD

$x(t) \approx UU^\top x(t) = Uy(t), \qquad y(t) = U^\top x(t).$

ROM takes the form:

$$\dot{y}(t) = U^\top AUy(t) + U^\top F(Uy(t)).$$

Nonlinear term still asks for the full vector

$$F(x) = \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_n(x_n) \end{bmatrix}.$$

POD+DEIM

Using $U^\top F(Uy(t)) \approx S_I^\top U^{-1} S_I^\top F(Uy(t))$.

ROM takes the form:

$$\dot{y}(t) = U^\top AUy(t) + (S_I^\top U)^{-1} S_I^\top F(Uy(t)).$$

Only requires

$$S_I^\top F(x) = \begin{bmatrix} f_{i_1}(x_{i_1}) \\ \vdots \\ f_{i_r}(x_{i_r}) \end{bmatrix}.$$

DEIM needs to evaluate only r entries instead of n entries of the high-dimensional function F.

TT-cross: matrix cross on unfoldings

What changes for tensors?

1: A tensor entry is indexed by several modes, e.g., $\mathcal{A}(i_1, \ldots, i_d)$.
2: Rows and columns are replaced by fibers, slabs, and interface matrices.
3: A tensor-cross method never forms all tensor entries.
4: It repeatedly builds small matrix-cross problems on unfoldings or interfaces.
5: Good-submatrix selection chooses stable fibers and interface indices.
6: The result is a compressed tensor representation, for example in tensor-train form.

[Oseledets/Tyrtyshnikov'10], [Savostyanov'14], [Goreinov/Tyrtyshnikov/Zamarashkin'97].

Conclusion

1. CUR/cross approximation is easy once the index sets are fixed.
2. The prototype separates the task into a space or column subset selection step, a good-submatrix step, and a core formation step.
3. The hard stability question is the conditioning of the selected submatrix.
4. LUPP, QRCP, `maxvol` are practical volume-based selectors.
5. Practical algorithms differ mainly in how they obtain the space approximator before applying a selector.
6. MOR-DEIM and tensor cross reuse the same good-submatrix principle in different contexts.

Thank you!